ON PRIME ENDS AND LOCAL CONNECTIVITY

Donald Sarason*

1. INTRODUCTION

Marie Torhorst [5] proved the following theorem in 1921: _Let G be a simply connected domain in the plane and P a prime end of G. Then the complement of G fails to be locally connected at all except possibly one or two points of the impression of P._

I should like to present here a proof of this result which brings out its connection with subsequent work of Ursell and Young [6]. These authors showed that the impression of a prime end breaks up into two "wings," and that there is a relation of "priority" among the points in either wing. The exceptional points in Torhorst's theorem, it turns out, are points of lowest priority.

In §2 I shall state the needed results of Ursell and Young in a form suitable for present purposes. Torhorst's theorem is proved in §3. The concluding §4 contains additional comments.

The reader is assumed familiar with the basic theory of prime ends as given in Chapter 9 of the book of Collingwood and Lohwater [3]. For the most part we follow the terminology and notation of [3].

It is regrettable that Torhorst's results have been overlooked by more recent workers in the theory of prime ends. Her paper is not mentioned in the book of Collingwood and Lohwater, nor in the paper of Ursell and Young, nor in the recent papers of Arsove [1], [2], where some of her results reappear.

2. THE RESULTS OF URSELL AND YOUNG

We consider a bounded, simply connected domain G and a prime end P of G. (The assumption that G is bounded involves no essential loss of generality.) The impression of P will be denoted by $I(P)$ and the set of principal points of P by $I_0(P)$.

By a _path_ in G we shall mean a continuous map of an interval on the line into G. However, we shall often speak of a path as if it were a point set in the plane; that is, we shall identify a path with its range. This will never cause confusion.

A path is called _simple_ if it is a one-to-one map. It is called _open_ or _half-open_ accordingly as its interval of definition is open or half open. A simple open path can be thought of as composed of two simple half-open paths meeting in a single point; we refer to the latter as the _right_ and _left halves_ of the open path.

Mainly we shall be concerned with half-open paths in G defined on $[0,1)$ and converging to the prime end P; these we call P-_paths_. If γ is a P-path, then the cluster set of γ, that is, the set $\bar{\gamma} - \gamma$, will be denoted by $C(\gamma)$.

We now choose in G a simple open path γ_0 such that (i) the right half of γ_0 is a principal P-path, and (ii) the left half of γ_0 is an endcut of G converging to a prime end distinct from P. We regard γ_0 as fixed throughout the discussion but reserve the right to choose its ~~[illegible]~~ left endpoint conveniently later on.

The set $G - \gamma_0$ is the union of two disjoint domains, which we denote by G_+ and G_-. A sequence in G is said to converge to P_+ if it converges to P and is eventually in G_+. The *positive wing* of P, denoted $I_+(P)$, is the set of all cluster points of sequences converging to P_+. The *negative wing* of P, denoted $I_-(P)$, is defined analogously. The sets $I_+(P)$ and $I_-(P)$ are compact connected subsets of $I(P)$. They contain $I_0(P)$, and their union is $I(P)$.

A P-path that is eventually in G_+ will be called a P_+-path. If w_1 and w_2 are points of $I_+(P)$, then w_1 is said to be *prior to* w_2 *modulo* P_+ if w_1 is contained in the cluster set of every P_+-path whose cluster set contains w_2. If w_1 is prior to w_2 modulo P_+ but w_2 is not prior to w_1 modulo P_+, then w_1 is said to be *strictly prior* to w_2 modulo P_+. If w_1 and w_2 are prior to each other modulo P_+, then they are said to have *equal priority* modulo P_+. When no confusion can arise, we shall drop the modifier "modulo P_+."

The relation of *priority modulo* P_- is defined analogously.

The basic theorem of Ursell and Young can be stated as follows.

THEOREM [6, p. 1]. _If_ γ_1 _and_ γ_2 _are_ P_+-_paths_, _then_ _either_ $C(\gamma_1) \subset C(\gamma_2)$ _or_ $C(\gamma_2) \subset C(\gamma_1)$.

From this theorem it follows that if the points w_1 and w_2 in $I_+(P)$ do not have equal priority modulo P_+, then one of them is strictly prior to the other. Thus the relation of priority modulo P_+ induces a linear order on the set of its equivalence classes.

A point of $I_+(P)$ is said to have _lowest priority_ in $I_+(P)$ if every other point of $I_+(P)$ is strictly prior to it modulo P_+. It may happen that $I_+(P)$ contains no point of lowest priority; this is the case, for example, when P is of the third kind, because all principal points of P have equal priority. There may fail to be a point of lowest priority for other reasons. For instance, it can happen that every point of $I_+(P)$ is strictly prior to some other point (see Figure 1 for an example).

The notions defined above for $I_+(P)$ apply equally to $I_-(P)$. Simple examples show that the following _a priori_ possibilities all occur:

(i) $I_+(P)$ and $I_-(P)$ both contain a point of lowest priority, and these point are distinct;

(ii) there is a single point which has lowest priority both in $I_+(P)$ and in $I_-(P)$;

(iii) one of $I_+(P)$ and $I_-(P)$ has a point of lowest priority and the other does not;

(iv) neither $I_+(P)$ nor $I_-(P)$ has a point of lowest priority.

If a point of lowest priority in $I_+(P)$ belongs to $I_-(P)$, it does not necessarily have lowest priority in $I_-(P)$.

If γ is a half-open path defined on the interval $[0,1)$, then by an ε-modification of γ (ε a positive number) we shall mean a path γ' defined on $[0,1)$ with the following properties: (i) there is a sequence $\{t_n\}_{n=1}^{\infty}$ in $[0,1)$ converging monotonically to 1 such that γ and γ' coincide in $[t_{2n}, t_{2n+1}]$ for all n; (ii) the set $\cup_1^{\infty} \gamma[t_{2n-1}, t_{2n}]$ has diameter less than ε. If γ is a P_+-path, then a modification of γ is called P_+-admissible if it, also, is a P_+-path.

Our proof of Torhorst's theorem is based on the following lemma.

LEMMA. *Let w be a point of $I_+(P)$ which is not of lowest priority in $I_+(P)$. Then there is a positive number ε with the following property: if γ is any P_+-path such that $C(\gamma) = I_+(P)$, then w belongs to $C(\gamma')$ for every P_+-admissible ε-modification γ' of γ.*

Under the hypotheses, there is a point w_1 distinct from w in $I_+(P)$ such that w is prior to w_1 modulo P_+. Let $\varepsilon = |w-w_1|$, and let γ and γ' be as in the statement of the lemma. Then w and w_1 belong to $C(\gamma)$, and the set $C(\gamma) - C(\gamma')$ has diameter less than ε. Hence $C(\gamma')$ must contain either w or w_1. But if $C(\gamma')$ contains w_1 then it also contains w, so the lemma follows.

3. TORHORST'S THEOREM

THEOREM. *Let w be a point in $I_+(P)$ at which the complement of G is locally connected. Then w has lowest priority in $I_+(P)$.*

Because each wing of P contains at most one point of lowest priority, this theorem implies the result of Torhorst stated in §1.

To prove the theorem, we may obviously assume that w is not the only point in $I_+(P)$. Let γ be a simple polygonal P_+-path(1) such that $C(\gamma) = I_+(P)$. We shall show that for every positive number ε, there is a P_+-admissible ε-modification γ' of γ such that w is not in $C(\gamma')$. In virtue of the above lemma, this will prove the theorem. We need only consider suitably small numbers ε, so we may assume that ε is less than the diameter of $I_+(P)$. Later we shall impose additional upper bounds on ε.

Let D denote the open disk with center w and radius $\varepsilon/2$. Then $I_+(P)$ contains both a point in D and a point outside of $\overline{D}$. Therefore the path γ does not lie either eventually outside of D or eventually inside D, so it has infinitely many intersections with ∂D. But because γ is polygonal, each subarc $\gamma[0,t]$, with $0 < t < 1$, has only finitely many intersections with ∂D. Therefore there is a sequence $\{t_n\}_{n=1}^{\infty}$ in $(0,1)$ tending monotonically to 1 such that $\gamma(t_{2n-1}, t_{2n})$ is contained in D for each n and $\gamma[t_{2n}, t_{2n+1}]$ is contained in the complement of D for each n. We denote the arc $\gamma(t_{2n-1}, t_{2n})$ by γ_n; this arc is a crosscut of D.

Let F denote the complement of G. Because F is locally connected at w, there is an open disk Δ centered at w and contained properly in D such that $\bar{\Delta} \cap F$ is contained in one component of $D \cap F$. To simplify matters slightly, we take Δ so that its boundary contains no vertices of γ. For each n such that γ_n meets Δ we modify γ in the interval $[t_{2n-1}, t_{2n}]$ as follows. Let D_n be the component of $D - \gamma_n$ that does not contain w. The intersection of D_n with $\partial\Delta$ is the union of finitely many disjoint open arcs $A_1, \cdots, A_m$. Let the endpoints of A_j be $\gamma(a_j)$ and $\gamma(b_j)$, where the notation is chosen so that $a_j < b_j$ and $a_1 < a_2 < \cdots < a_m$. It is [redacted] easily seen that, of the two points a_j and b_j, one must correspond to a point of entry of γ into Δ and the other to a point of exit of γ from Δ. Now D_n does not meet the component of $D \cap F$ that contains w, and hence $D_n \cap \bar{\Delta}$ does not meet F. The arcs A_j are therefore contained in G. We first modify γ by replacing $\gamma(a_1, b_1)$ by the arc A_1, suitably parameterized. If b_1 exceeds every other b_j we make no further modification of γ in $[t_{2n-1}, t_{2n}]$. Otherwise, as is easily seen, the next intersection after $\gamma(b_1)$ of γ with $\partial\Delta$ is one of the points $\gamma(a_j)$, say $\gamma(a_k)$, and we replace $\gamma(a_k, b_k)$ by the arc A_k. If b_k exceeds every other b_j we make no further modification of γ in $[t_{2n-1}, t_{2n}]$. Otherwise the next intersection after $\gamma(b_k)$ of γ with $\partial\Delta$ is one of the points $\gamma(a_j)$, and we proceed as before. After finitely many such steps we produce a modification of γ in the interval $[t_{2n-1}, t_{2n}]$ that remains outside the disk Δ. (Although some of the topological points we have glossed over here are not completely trivial, the reader should be able to dispose of them easily by using simple properties of Jordan curves.)

The above procedure, carried out for each n, yields an ε-modification γ' of γ which lies in G and whose cluster set does not contain w. It remains to show that γ' is a P_+-path, at least if ε is sufficiently small.

Let $\{\alpha_i\}_{i=1}^{\infty}$ be a chain of crosscuts of G belonging to the prime end P and converging to a point w_0. The paths γ and γ' have common points arbitrarily close to ∂G. Because γ converges to P, the path γ' therefore at least contains a sequence of points converging to P. This implies that γ' meets all except possibly finitely many of the crosscuts α_i. Hence w_0 is in $C(\gamma')$, and therefore $w_0 \neq w$.

We assume now that $\varepsilon < |w-w_0|$. Then for i large enough, the crosscut α_i is disjoint from the disk D, so that in modifying γ to γ' we neither create nor destroy intersections with α_i. It follows that γ' converges to P.

Because w is not in $C(\gamma')$, we may now conclude that w is not a principal point of P. Hence w is at a positive distance from γ_0, provided we assume γ_0 was chosen at the onset so that its left endpoint is distinct from w. We assume, finally, that $\varepsilon < \text{dist}(w, \gamma_0)$. Then in modifying γ to γ' we neither create nor destroy intersections with γ_0, and therefore γ' must, with γ, lie eventually in G_+. Hence γ' is a

P

P_+-path, as desired.

The proof of the theorem is complete.

4. COMMENTS

(4.1) Torhorst originally stated her conclusions in terms of the boundary of G rather than the complement of G. It was pointed out by Hahn [4], however, that the theorem of §1 can be obtained by a trivial modification of her arguments. This theorem implies that if the complement of G is locally connected, then all prime ends of G are of the first kind, so that, by the Carathéodory mapping theorem, any conformal map of the open unit disk onto G extends to a continuous map of the closed disk onto $\overline{G}$. If the latter happens then the boundary of G is a continuous image of the unit circle and so is locally connected. Hence, <u>if the complement of a bounded simply connected domain is locally connected then the boundary of the domain is also locally connected</u>. (The converse is trivial.)

The last theorem, formulated slightly differently, is what point set topologists call Torhorst's theorem; see for example [7, p. 106].

(4.2) The following theorem is a local converse to the statement that a bounded simply connected domain with a locally connected boundary has only prime ends of the first kind.

THEOREM. Let w be a point of ∂G which is contained in the impressions of prime ends of the first kind only. Then ∂G is locally connected at w.

To prove this, let D_0 denote the open unit disk, and let f be a conformal map of D_0 onto G. For S a subset of ∂D_0 we let $C(f,S)$ denote the cluster set of f at S.

Let E be the set of points z in ∂D_0 such that $C(f,z)$ contains w (and therefore equals $\{w\}$). Then E is compact and f is continuous at each point of E. Hence, if D is any open disk centered at w, there is a relatively open subset J of ∂D_0 such that $E \subset J$ and $C(f,J) \subset D$. We may assume that each component of J meets E. If I is one of these components, then $C(f,I)$ is a connected subset of ∂G containing w. It follows that $C(f,J)$ is a connected subset of ∂G, and hence $C(f,J)$ is contained in a single component of $\partial G \cap D$. Therefore $\partial G - C(f, \partial D_0 - J)$ is a relatively open subset of ∂G containing w and contained in a single component of $\partial G \cap D$. We may conclude that ∂G is locally connected at w, as desired.

It follows trivially that the complement of G is also locally connected at w.

(4.3) It is possible for $I_+(P)$ to contain a point w of lowest priority, such that w is not in $I_-(P)$ nor in the impression of any prime end besides P, but such that the complement of G is not locally connected at w. An example is given in Figure 2.

(4.4) Torhorst proved the theorem of §1 in two steps. First she showed that if ∂G is locally connected at the point w of I(P), then ∂G is locally connected at w from one side of P (see [5] for the definition of the latter property). We shall call a point of I(P) at which ∂G is locally connected from one side of P a Torhorst point of P. The second step in her proof was to show that P has at most two Torhorst points.

There is no simple relation between the notion of a Torhorst point and that of a point of lowest priority. In the domain of Figure 3, there are two prime ends whose impression is the thick horizontal segment, and the point w has lowest priority in the nondegenerate wing of one of these prime ends. However, w is not a Torhorst point of this prime end. For the prime end of Figure 1, the point w is a Torhorst point, but it has highest priority in both wings of the prime end.

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FOOTNOTES

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1. In other words, we require $\underline{\gamma}$ to be a P_+-path such that $\underline{\gamma}[0,t]$ is a polygonal Jordan arc for each t in $(0,1)$. A straightforward construction produces such a $\underline{\gamma}$ that in addition satisfies $C(\underline{\gamma}) = I_+(P)$.

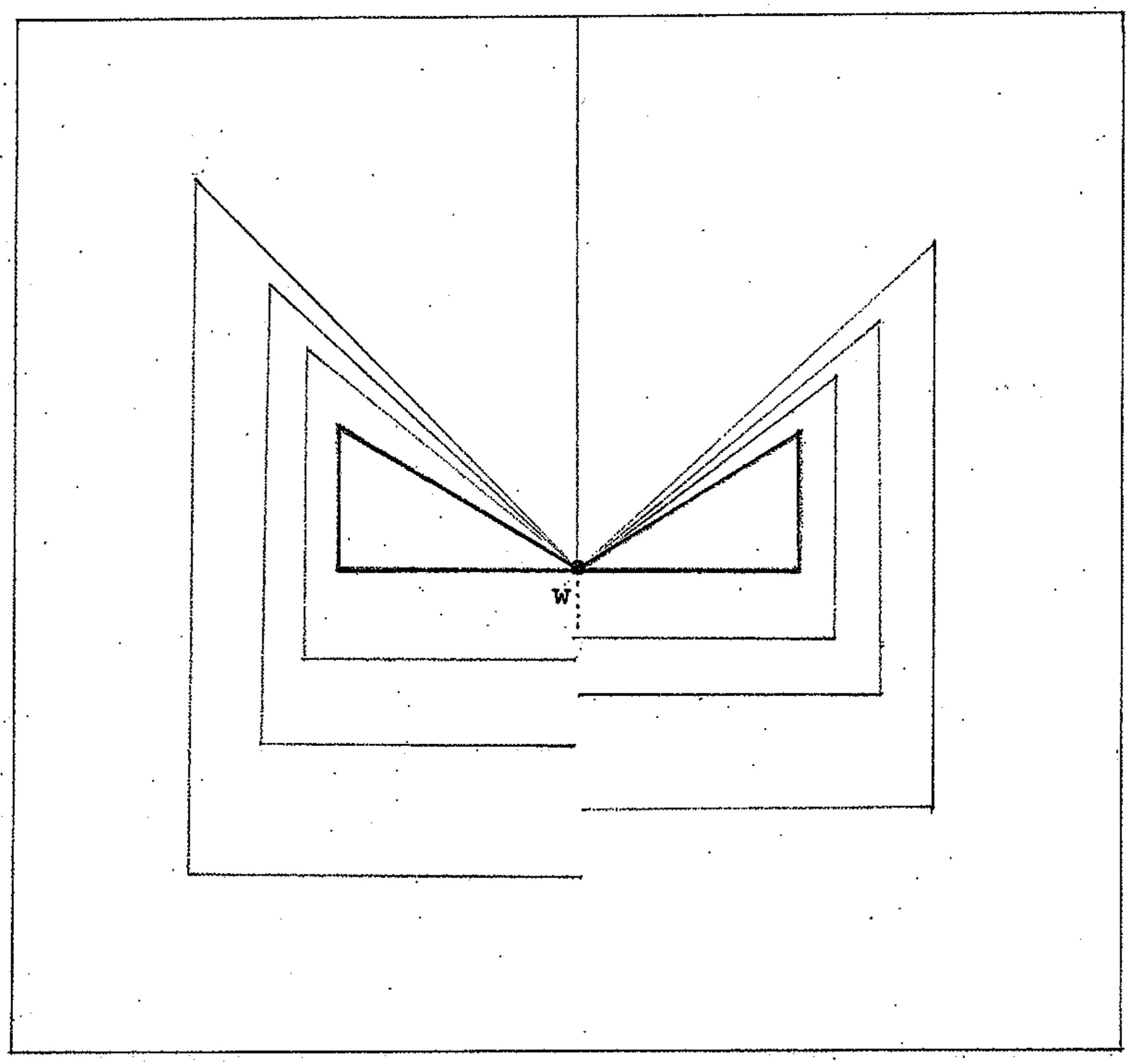

FIGURE 1

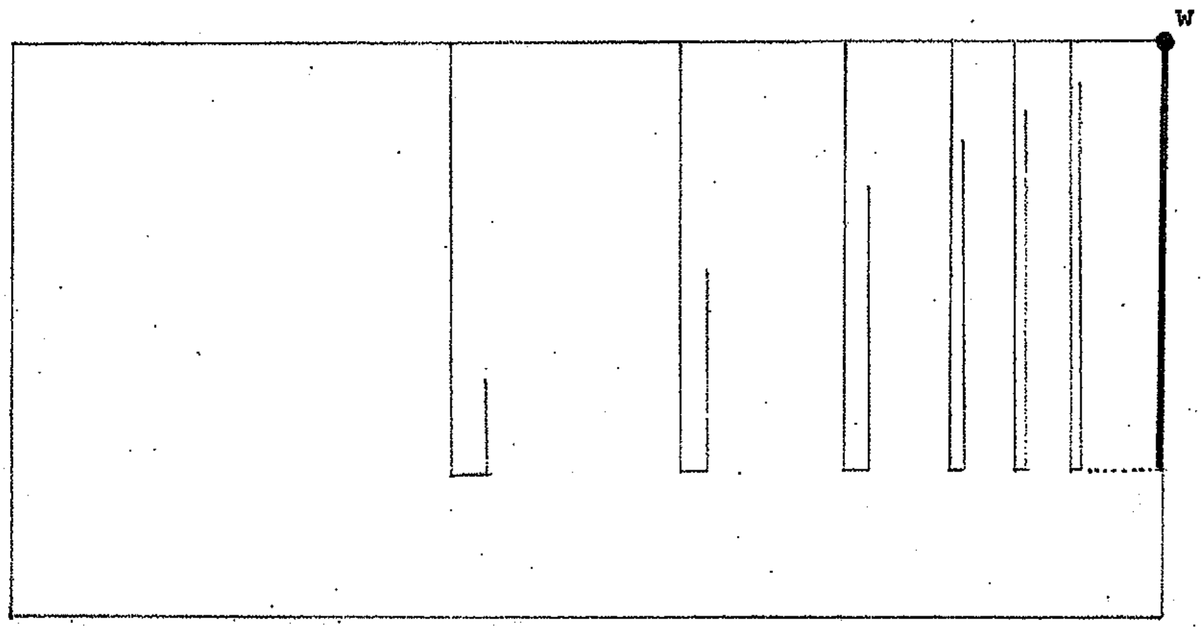

FIGURE 2

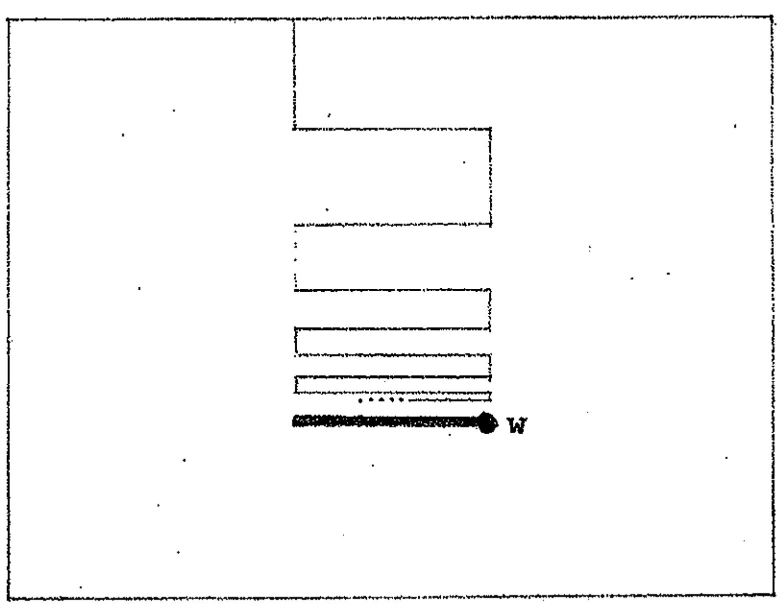

FIGURE 3

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AREA CODE 313
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October 15, 1968

Professor Donald Sarason
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Dear Donald,

When Collingwood was here, we discussed your paper on prime ends and local connectivity.

At one time, Collingwood and I were struggling with prime ends; we looked at Marie Torhorst's article in the Zeitschrift, but for some reason we decided against working it into the list of references in our paper.

Collingwood and I were concerned over the failure of Carathéodory's definitions to pay homage to points that play multiple roles in the impression of one prime end. We knew that the prime ends of the second and fourth kinds are always scarce enough so that they form a set of the first category, and we tried desperately to prove that the same is true of the prime ends whose impressions have points that behave as if they wanted to rebel against their suburban status of principal points. To our enormous astonishment, we found an example in which every prime end is either of the second kind, or else is of the third kind but raises a false claim of misclassification.

At the time, it was not yet known how bad the prime ends of a domain can look. Of course, there was Denjoy's example; but partly because mathematical libraries usually don't stock the C. R. Acad. Sci. Paris, and partly because of Denjoy's awkard description, the example had attracted little attention and had been forgotten by almost everybody.

The book of evils that can afflict a prime end will never be completely closed; however, I believe that interest has waned so much that while a man might write another page in the book, hardly anybody would read it. I suggest that you put your stuff on ice until either

α) you have an application for it, or

β) you can incorporate it into a treatise.

In case you think that I am crazy, I agree, but point out that neither you nor I can do much about it.

With cordial greetings,

George Piranian

GP/san

Enclosure

critically n-connected if $\varkappa(G) = n$ and $\varkappa(G - v) = n - 1$ for each point v of G. Let $\mu(G)$ denote the
minimum degree of G. The inequality $\varkappa(G) \leqq \mu(G)$ is well known. The authors have proved the
following theorem. Theorem. There exists no critically n-connected graph with $\mu(G) \geqq (3n - 1)/2$.
The authors have constructed examples to show that this inequality is best possible. (Received
March 10, 1969.)

69T-H34. WALTER ALLEGRETTO, University of British Columbia, Vancouver 8, B.C.,
Canada. Comparison and oscillation theorems for elliptic operators. Preliminary report.

Let λ_0 denote the smallest real eigenvalue in a smooth bounded domain G of the uniformly
elliptic operator L, defined by $Lu = -\Sigma D_i(a_{ij}D_ju) + \Sigma b_jD_ju + cu$, with sufficiently smooth
coefficients and vanishing boundary conditions. Let $B = (b_1/2, \ldots, b_n/2)$, $h = -\Sigma b_i b_i^*/2 \det(a_{ij})$
where b_i^* denotes the cofactor of $b_i/2$ in the matrix $\left(\begin{array}{c|c}(a_{ij}) & B^T \\ \hline B & h\end{array}\right)$. Theorem 1. Let μ denote the smallest
eigenvalue of $[(L + L^*)/2] + h$ in a smooth bounded domain D such that $D \subset G$, then $\lambda_0 < \mu$.
Here L^* denotes the formal adjoint of L. Theorem 2. $\lambda_0 \geqq \mu_1$ where μ_1 denotes the smallest
eigenvalue of $(L + L^*)/2$ for G. A sufficient condition for $\lambda_0 > \mu_1$ is that $\Sigma D_i(b_i)$ never vanish in G.
The monotonicity and continuity of λ_0 as a function of the domain are considered. Several strong
oscillation theorems are an immediate consequence of the above results. (Received March 31, 1969.)

69T-H35. DONALD E. SARASON, University of California, Berkeley, California 94720.
On prime ends and local connectivity.

Marie Torhorst [Über den Rand der einfach zusammenhängenden ebenen Gebiete, Math. Z. 9
(1921), 44-65] has proved the following Theorem. If G is a simply connected domain in the plane and
is a prime end of G, then ∂G fails to be locally connected at all except possibly one or two
oints of I(P) (the impression of P). This paper gives a new proof of Torhorst's theorem, based on
ork of Ursell and Young [Remarks on the theory of prime ends, Memoirs Amer. Math. Soc. No. 3,
951]. These authors showed that the impression of a prime end breaks up into two "wings," and
at there is a relation of "priority" among the points in either wing. The new proof establishes
at any point of I(P) at which ∂G is locally connected is a point of lowest priority. (Received
rch 26, 1969.)

69T-H36. SHEILA A. GREIBACH, Harvard University, Cambridge, Massachusetts 02138.
ins of full AFL's.

Definitions. For full AFL's $\mathscr{L}_1, \mathscr{L}_2$, let $\mathscr{L}_1 \hat{\sigma} \mathscr{L}_2$ be the family of all $\tau(L)$, $L \subseteq \Sigma_1^*$ in $\mathscr{L}_1$ and
substitution with $\tau(a)$ in $\mathscr{L}_2$ for a in Σ_1. Let $\hat{\mathscr{S}}(\mathscr{L})$ be the substitution closure of $\mathscr{L}$. Let
$) = aL$ for a in Σ. Let $\mathscr{R}$ be the regular sets, and $\mathscr{C}$ the context-free. Lemma. Let $L_1 \subseteq \Sigma_1^+$,
Σ_2^+, $\Sigma_1 \cap \Sigma_2 = \emptyset$. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be full AFL's. (1) If $\tau_{L_2}^{\Sigma_1}(L_1) \in \mathscr{L}_1 \hat{\sigma} \mathscr{L}_2$, either $L_1 \in \mathscr{L}_1$ or
$\mathscr{L}_2$. (2) If $\tau_{L_2}^{\Sigma_1}(L_1) \in \mathscr{R} \hat{\sigma} (\mathscr{L}_1 \cup \mathscr{L}_2)$, either L_1 is in $\mathscr{L}_1$ or L_2 is in $\mathscr{L}_2$. Theorem. If $\mathscr{L}$ is a
bstitution closed full AFL, so is $\mathscr{L} \hat{\sigma} \mathscr{L}$, and $\hat{\mathscr{S}}(\mathscr{L})$ is not full principal. Theorem. If $\mathscr{L}_1$ and $\mathscr{L}_2$
comparable full AFL's, then $\hat{\mathscr{F}}(\mathscr{L}_1 \cup \mathscr{L}_2)$ is not substitution closed and $\mathscr{L}_1 \hat{\sigma} \mathscr{L}_2$ and $\mathscr{L}_2 \hat{\sigma} \mathscr{L}_1$
comparable and $\hat{\mathscr{S}}(\mathscr{L}_1 \cup \mathscr{L}_2)$ is not full principal. Corollary. If $\mathscr{L} \subseteq \mathscr{C}$ is a nonsubstitution
full AFL, $\hat{\mathscr{S}}(\mathscr{L}) \neq \mathscr{C}$. Corollary. If Q is a family of languages with $\mathscr{F}(Q) = \mathscr{C}$, then $\mathscr{C} = \hat{\mathscr{F}}(L)$ for